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THE THEORY OF DUALITY IN SYSTEMS WITH AFTEREFFECT*

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Some problems of control and observation /1—3/ of linear dynamic systems with aftereffect, defined by differential and integral equations with deviating arguments are considered. The theory of duality for the problem of minimizing the Boltz convex functional on the trajectories of a functionally differentiable system of the neutral type with a lag in the control, state, and velocity variables is developed. New concepts of controllability are introduced into the system with aftereffects and phase constraints, as well as dual concepts of ideal observability of their conjugate system of integral equations with a lead in conditions of incomplete information. The observability concepts introduced here are connected with the restitution of the generalized final state of the system containing minimum information to enable the future motion to be calculated uniquely. The schemes and results obtained enable them to be used in differential-game problems of dynamic systems with aftereffects /4—6/.

1. The problem of optimal control. Consider a linear control system whose dynamics along the segment $[t_0, t_1]$ is defined by differential equations with a deflecting argument of the neutral type

$$x^{\cdot}(t) = A(t)x(t) + A_1(t)x(t-h) + A_2(t)x^{\cdot}(t-h) + B(t)u(t) + B_1(t)u(t-h) \tag{1.1}$$

where $h > 0$ is the lag of the control, state and velocity variables.

Systems with an aftereffect of the type (1.1) occur in problems of mechanics, automatic control, economics, etc. (see the numerous examples in /7/). It is important to allow for the action of the aftereffect when defining real dynamic systems and related control and observation processes.

Let us consider the problem of minimizing the Boltz functional

$$I(x, u) = \Phi(x(t_0), x(t_1)) + \int_{t_0}^{t_1} F(x(t), u(t), t)\,dt \to \inf \tag{1.2}$$

*Prikl.Matem.Mekhan., 48, 4, 622-631, 1984 on a set of absolutely continuous trajectories $x: [t_0, t_1] \to R^n$ and summable controls $u: [t_0, t_1] \to R^m$ that satisfy system (1.1) with initial conditions and constraints

$$x(t) = \varphi(t), \quad t_0 - h \leqslant t < t_0; \quad u(t) = \varphi_1(t), \quad t_0 - h \leqslant t < t_0 \tag{1.3}$$

$$(x(t_0),\ x(t_1)) \in D \subset R^{2n}; \quad u(t) \in U(t) \subset R^m, \quad t_0 \leqslant t \leqslant t_1 \tag{1.4}$$

Henceforth we will assume that the following conditions, imposed on parameters of problem (1.1)−(1.4), are satisfied:

a) the function $\Phi: R^{2n} \to (-\infty, \infty]$ is convex and semicontinuous from below, and

$$\operatorname{dom} \Phi \overset{\text{def}}{=} \{(x_0, x_1) \in R^{2n}: \Phi(x_0, x_1) < \infty\} = D$$

b) the function $F(x, u, t)$ is convex relative to (x, u), measurable and essentially bounded with respect to t, and $\operatorname{dom} F(\cdot, \cdot, t) = R^n \times R^m$ for almost all $t \in [t_0, t_1]$;

c) the multivalued mapping $U: [t_0, t_1] \to 2^{R^m}$ is measurable /3, 8/ and takes closed convex values, and the function $d(t) = \inf\{|u| : u \in U(t)\}$ is essentially bounded along $[t_0, t_1]$;

d) the components of the $(n \times n)$-matrices $A(t), A_1(t)$ are summable, while the components of the $(n \times n)$-matrix $A_2(t)$ and of the $(n \times m)$-matrices $B(t)$ and $B_1(t)$ are measurable and are essentially bounded along $[t_0, t_1]$;

e) the vector function $\varphi: [t_0 - h, t_0] \to R^n$ is absolutely continuous, and the vector function $\varphi_1: [t_0 - h, t_0] \to R^m$ is summable;

f) the following regularity conditions of the Slater type are satisfied: a process $\{x(t), u(t)\}$, $t_0 \leqslant t \leqslant t_1$ exists that is admissible in (1.1)−(1.4) and satisfies the inclusions

$$(x(t_0), x(t_1)) \in \operatorname{ri} D; \quad u(t) \in \operatorname{ri} U(t), \quad t_0 \leqslant t \leqslant t_1 \tag{1.5}$$

where $\operatorname{ri} X$ denotes the relative interior of the set X /9/. If the set

$$\operatorname{epi} \Phi \overset{\text{def}}{=} \{(x_0, x_1, \mu) \in R^{2n+1}: \mu \geqslant \Phi(x_0, x_1),\ (x_0, x_1) \in D\}$$

is polyhedral /9/, the first inclusion in (1.5) may be weakened to $(x(t_0), x(t_1)) \in D$. Note that condition f) is satisfied a fortiori, if the right-hand end of $x(t_1)$ is free of constraints.

Henceforth everywhere the prime denotes transposition, $|\cdot|$ is the norm in a finite dimensional space, and $\delta(\cdot \mid X)$ is the indicator function, of the set X.

2. The dual problem and the conditions of optimality.

We consider the conjugate functions /9/

$$\Phi^*(\psi_0, \psi_1) = \sup\{x_0'\psi_0 + x_1'\psi_1 - \Phi(x_0, x_1): (x_0, x_1) \in R^{2n}\}$$
$$F^*(w, y, t) = \sup\{u'y + x'w - F(x, u, t): u \in U(t),\ x \in R^n\}$$

and we construct the problem dual to (1.1)−(1.4) of minimizing the functional

$$\begin{aligned} J(\psi, w) = {} & \int_{t_0}^{t_0+h} \psi'(t)[A_1(t)\varphi(t-h) + A_2(t)\varphi^{\cdot}(t-h) + \\ & B_1(t)\varphi_1(t-h)]\,dt + \int_{t_0}^{t_1} F_0^*(w(t), B'(t)\psi(t) + \\ & B_1'(t+h)\psi(t+h), t)\,dt + \Phi^*(\psi(t_0) - A_2'(t_0+h)\psi(t_0+h), \\ & -\psi(t_1)) \to \inf \end{aligned} \tag{2.1}$$

on the set of summable controls $w: [t_0, t_1] \to R^n$ and essentially bounded trajectories $\psi: [t_0, t_1] \to R^n$ of the integral system with the lead

$$\begin{aligned} \psi(t) = {} & \psi(t_1) + \int_t^{t_1} [A'(\tau)\psi(\tau) + A_1'(\tau+h)\psi(t+h) - w(\tau)]\,dt + \\ & A_2'(t+h)\psi(t+h), \quad t_0 \leqslant t \leqslant t_1; \quad \psi(t) \equiv 0, \quad t_1 < t \leqslant t_1 + h \end{aligned} \tag{2.2}$$

Note that by virtue of (2.2), the function $\psi(t) - A_2'(t+h)\psi(t+h)$ is absolutely continuous on $[t_0, t_1]$, and the condition of finiteness of the functional (2.1) results in contraints expressed in terms of effective sets of conjugate functions Φ^*, F_0^* /9/

$$(\psi(t_0) - A_2'(t_0+h)\psi(t_0+h),\ -\psi(t_1)) \in \operatorname{dom} \Phi^* \tag{2.3}$$

$$w(t) \in \operatorname{dom} F_0^*(\cdot, B'(t)\psi(t) + B_1'(t+h)\psi(t+h), t),\ t_0 \leqslant t \leqslant t_1 \tag{2.4}$$

System (2.2) represents the totality of integral equations of the Volterra type of the second kind with a leading argument. If the matrix $A_2(t)$ is absolutely continuous on $[t_0, t_1]$, then by virtue of (2.2) we obtain that the trajectory $\psi(t)$ is piecewise continuous on $[t_0, t_1]$, the points of discontinuity are of the form $\tau_i = t_1 - ih,\ i = 1, 2, \ldots,$ and $\psi(t)$ is absolutely continuous over every interval of continuity. Differentiating (2.2), we obtain in this case the equivalent system of functionally differential equations with a lead of neutral type

$$\psi^{\cdot}(t) = -A'(t)\psi(t) - A_1'(t+h)\psi(t+h) + d/dt\,[A_2'(t+h)\psi(t+h)] + w(t) \tag{2.5}$$
$$t_0 \leqslant t \leqslant t_1;\quad \psi(t) \equiv 0,\quad t_1 < t \leqslant t_1 + h$$

with the conditions of trajectory jumps

$$\psi(\tau_i - 0) - \psi(\tau_i + 0) = A_2'(\tau_i + h)[\psi(\tau_i + h - 0) - \psi(\tau_i + h + 0)] + \Big[\prod_{k=1}^{i} A_2'(t_1 + (k-t)h]\psi(t_1),\quad \tau_i = t_1 - ih,\quad i = 1, 2, \ldots \tag{2.6}$$

from which it follows that the trajectory $\psi(t)$, $t_0 \leqslant t \leqslant t_1$ does not have jumps, if either $A_2(t_1) = 0$, or the boundary condition (2.3) implies $\psi(t_1) = 0$.

Note that the control set in (2.4) does not depend on the state variables of system (2.2), i.e. problem (2.1)–(2.4) has no phase constraints, if $F(x, u, t) = F_1(x, t) + F_2(u, t)$ and the function $F_2(u, t) + \delta(u|U(t))$ is cofinite with u /9/ for almost all $t \in [t_0, t_1]$. This occurs if the given function satisfies the condition of growth of the Nagumo-Tonelli type as $|u| \to \infty$ /8/, in particular, when the set $U(t)$ is uniformly bounded, or $F_2(\cdot, t)$ increases at infinity more rapidly than $|u|$.

The following result establishes the duality relation between the extremal values of the functionals in problems (1.1)–(1.4) and (2.1)–(2.4) and the necessary and sufficient conditions related to it. In the relations defined below $\partial\Phi(x_0, x_1)$ denotes the subdifferential of the function $\Phi(\cdot,\cdot)$ /9/ in the sense of convex analysis at the point (x_0, x_1), and $\partial_x F(x, u, t)$ denotes the subdifferential of the function F with respect to the first argument. Note that the subdifferential multivalued mappings take convex closed values and, in the case of smooth convex functions, reduce to conventional derivatives.

Theorem 2.1. When assumptions a)-f) hold in problem (2.1)–(2.4), a solution exists and the extremal relation of duality

$$\inf I(x, u) = -\min J(\psi, w) < \infty \tag{2.7}$$

holds. In this problem **inf** and **min** are taken over all admissible processes in problems (1.1)–(1.4) and (2.1)–(2.4), respectively. For the process $\{x^\circ(t), u^\circ(t)\}$, $t_0 \leqslant t \leqslant t_1$, to be optimal it is necessary for problem (1.1)–(1.4), and for almost all $F(x, u, t) = F_1(x, t) + F_2(u, t)$ it is also sufficient, that almost all $t \in [t_0, t_1]$ the following conditions be satisfied:

$$[(\psi^\circ(t))' B(t) + (\psi^\circ(t+h))' B_1(t+h)]\, u^\circ(t) - F(x^\circ(t), u^\circ(t), t) = \sup\{[(\psi^\circ(t))' B(t) + (\psi^\circ(t+h))' B_1(t+h)]\, u - F(x^\circ(t), u, t):\ u \in U(t)\} \tag{2.8}$$

$$w^\circ(t) \in \partial_x F(x^\circ(t), u^\circ(t), t) \tag{2.9}$$

$$(\psi^\circ(t_0) - A_2'(t_0 + h)\psi^\circ(t_0 + h),\ -\psi^\circ(t_1)) \in \partial\Phi(x^\circ(t_0), x^\circ(t_1)) \tag{2.10}$$

where $\{\psi^\circ(t), w^\circ(t)\}$, $t_0 \leqslant t \leqslant t_1$ is the optimal process in the dual problem (2.1)–(2.4).

Proof. In conformity with the scheme * we reduce the initial problem of optimal control (1.1)–(1.4) for the system with aftereffect to that of minimizing the Boltz convex functional on trajectories of a linear system of ordinary differential equations.

Let N be a positive integer such that

$$(N-1)h < t_1 - t_0 \leqslant Nh$$

Let us consider the vector functions $p(t)$ and $\nu(t)$ of dimensions Nn and Nm, respectively, on the segment $[0, h]$

$$p(t) = (p_1(t), \ldots, p_N(t)),\quad p_i(t) = x(t_0 + t + (i-1)h) \tag{2.11}$$
$$\nu(t) = (\nu_1(t), \ldots, \nu_N(t)),\quad \nu_i(t) = u(t_0 + t + (i-1)h),\ i = 1, \ldots, N$$

It can be shown that (1.1) is equivalent to the following set of ordinary differential equations in functions (2.11):

$$p^{\cdot}(t) = M^{-1}(t) K(t) p(t) + M^{-1}(t) L(t) \nu(t) + M^{-1}(t) g(t),\quad 0 \leqslant t \leqslant h \tag{2.12}$$
$$K(t) = (K_{ij}(t)),\quad L(t) = (L_{ij}(t)),\quad M(t) = (M_{ij}(t)),\quad 1 \leqslant i,\ j \leqslant N$$

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$$K_{ij}(t)=\begin{cases}A(t_0+t+(i-1)h), & j=i, \quad i=1,\ldots,N\\ A_1(t_0+t+(i-1)h), & j=i-1, \quad i=2,\ldots,N\\ 0 & \text{for all other} \quad i,j\end{cases}$$

$$L_{ij}(t)=\begin{cases}B(t_0+t+(i-1)h), & j=i, \quad i=1,\ldots,N\\ B_1(t_0+t+(i-1)h), & j=i-1, \quad i=2,\ldots,N\\ 0 & \text{for all other} \quad i,j\end{cases}$$

$$M_{ij}(t)=\begin{cases}E_n, & j=i, \quad i=1,\ldots,N\\ -A_2(t_0+t+(i-1)h), & j=i-1, \quad i=2,\ldots,N\\ 0 & \text{for all other} \quad i,j\end{cases}$$

and $g(t)=(f(t),0,\ldots,0), f(t)=A_1(t_0+t)\varphi(t_0+t-h)+B_1(t_0+t)\varphi_1(t_0+t-h)+A_2(t_0+t)\varphi^{\cdot}(t_0+t-h)$, where E_n is a unit matrix of dimensions $n\times n$.

The Boltz functional (1.2) and the constraints (1.4) take the form

$$\Phi(p_1(0),p_N(h))+\int_0^h\sum_{i=1}^N F(p_i(t),\nu_i(t),t_0+t+(i-1)h)\,dt\to\inf \tag{2.13}$$

$$(p_1(0),\ p_N(h))\in D,\quad p_i(h)=p_{i+1}(0),\quad i=1,\ldots,N-1 \tag{2.14}$$
$$\nu(t)\in U(t_0+t)\times U(t_0+t+h)\times\ldots\times U(t_0+t+(N-1)h),\quad 0\leqslant t\leqslant h$$

We apply to the problem of optimal control (2.12)−(2.14) the results obtained in /10/. These are, in turn, based on the reduction to the Boltz generalized variational problem /8/ and on the use of the theory of duality for problems of convex programming in functional space. Using the specific properties of problem (2.12)−(2.14) and passing from the hybrid system with a lead, conjugate to (2.12), (1.1), to the integral form (2.2), we obtain the duality problem of optimization in the form (2.1)−(2.4) and the extremal duality relation (2.7) from which, following the scheme usually applied to convex problems, we derive the necessary and sufficient conditions (2.8)−(2.10). The theorem is proved.

Remarks. 2.1. Relations (2.8)−(2.10) are sufficient for $\{x^\circ(t),u^\circ(t)\}$, $t_0\leqslant t\leqslant t_1$ to be optimal in problem (1.1)−(1.4) without the condition of regularity f).

2.2. The conditions of optimality (2.8)−(2.10) are an analogue of the Kuehn-Thakker theorem of convex programming /9/ for the class considered here of linearly convex problems of optimizing systems with aftereffect. When the functions F,Φ are smooth, the version of Pontryagin's principle of the maximum /11/ (in standard form) is strengthened, and the conjugate trajectory is the solution of the dual optimization problem. Krasovskii /1/ was the first to establish the results of this type for the special class of linearly convex problems of optimal control, and later developed in /1−4, 8, 10, 12/ and elsewhere for problems of control of ordinary dynamic systems and for systems with a lag.

2.3. By analogy with /8/ for ordinary systems the conditions of optimality obtained may be formulated in Lagrangian and Hamiltonian forms equivalent to (2.8)−(2.10) and, also, in the form of the theorem on the saddle point /1, 3, 4/ using the formalism of the theory of games. Note that for the special class of functions F,Φ the duality problem (2.1)−(2.4) can be interpreted in the form of the problem of optimal observation from the type of construction /1, 3, 13, 14/.

3. Controllability and observability. Using the theory of duality of extremal convex problems and the results obtained above, we shall consider some concepts of the controllability of linear systems with aftereffects of the type (1.1) and (2.2). We shall cnsider the control system (1.1) on the segment $[t_0,t_1]$ with the initial conditions

$$x(t)=\varphi(t),\quad u(t)\equiv 0,\quad t_0-h\leqslant t\leqslant t_0 \tag{3.1}$$

where $\varphi\colon[t_0-h,t_0]\to R^n$ is an absolutely constinuous vector function. We assume the vector functions $u(\cdot)\in L_\infty([t_0,t_1],R^m)$ to be admissible controls in (1.1) and, also, that the parameters of system (1.1) satisfy condition d).

Let $C(t)$ be a $(k\times n)$-matrix and $D(t)$ be a $(k\times m)$-matrix with elements that are measurable and essentially constrained on $[t_0,t_1]$ We introduce the constraints

$$\alpha_{x,u}(t)\overset{\text{def}}{=}C(t)x(t)+D(t)u(t)\equiv 0,\quad t_0\leqslant t\leqslant t_1 \tag{3.2}$$

on the admissible processes of system (1.1), (3.1) pertaining to constraints of mixed type on the phase coordinates and control.

We denote by $x_t(\cdot)$ the set $\{x(t);x(t+\theta),-h\leqslant\theta<0\}$, which we shall call the state of system (1.1) at the instant of time t. Let Ω be an arbitrary Banach space with the norm $\|\cdot\|_\Omega$ containing the final state $x_{t_1}(\cdot)$ of system (1.1), and P be the Banach space with the norm $\|\cdot\|_P$ that contains the functions $\alpha_{x,u}(\cdot)$ of the form (3.2) generated by processes $\{x(t),u(t)\}$, $t_0\leqslant t\leqslant t_1$ admissible in (1.1).

Definition 3.1. System (1.1), (3.1) is called Ω-approximately null-controllable on the segment $[t_0, t_1]$, when the constraints (3.2) are P approximations, it for any initial state $x_{t_0}(\cdot)$ over any $\varepsilon > 0$ there is a control $u(\cdot) \in L_\infty([t_0, t_1], R^m)$ and, by virtue of (1.1), (3.1), a trajectory $x(t), t_0 \leqslant t \leqslant t_1$ corresponding to it such that

$$\| x_{t_1}(\cdot) \|_\Omega + \| \alpha_{x,u}(\cdot) \|_P \leqslant \varepsilon$$

Remarks. 3.1. If $\| x_{t_1}(\cdot) \|_\Omega = | x(t_1) |$ and there are no constraints (3.2), the approximate null-control of system (1.1), (3.1) is identical with the exact controllability to null when $t = t_1$. However, in the general infinite - dimensional case the concepts of approximate and exact (or complete in the meaning of /15/) null controllability are not equivalent in the class of admissible controls consdidered here.

3.2. The vector function $\alpha_{x,u}(t)$ in (3.2) may be considered as the output of system (1.1) /3, 10/, and we can interpret the concept of controllability as the approximate null controllability of system (1.1), (3.1) with respect to the output of (3.2).

Let us consider further the system of observation, conjugate to (1.1), (3.1)

$$\psi(t) = \psi(t_1) + \int_t^{t_1} [A'(\tau)\psi(\tau) + A_1'(\tau + h)\psi(\tau + h)]\, d\tau + \tag{3.3}$$

$$A_2'(t + h)\psi(t + h) + \int_t^{t_1} [\gamma(\tau) - C'(\tau) v(\tau)]\, d\tau, \quad t_0 \leqslant t \leqslant t_1$$

$$\psi(t) \equiv 0, \quad t_1 \leqslant t \leqslant t_1 + h; \quad \gamma(t) \equiv 0, \quad t_0 \leqslant t \leqslant t_1 - h$$

$$z(t) = B'(t)\psi(t) + B_1'(t + h)\psi(t + h) - D'(t) v(t), \quad t_0 \leqslant t \leqslant t_1 \tag{3.4}$$

in which the aftereffect (the lead) reveals itself in the equations of the object of observation (3.3) and of the measuring equipment (3.4).

Note that m-dimensional output quantity $z(t)$, which can be measured, depends on the n-dimensional trajectory $\psi(t)$ of system (3.3) and, also, on the k-dimensional undefined perturbation $v(t)$, $t_0 \leqslant t \leqslant t_1$ that generates it.

The quantity $\psi_{t_1}(\cdot) = \{\psi(t_1); \gamma(t), t_1 - h \leqslant t \leqslant t_1\}$ plays the part of the initial state of system (3.3) with lead, which for any $\psi(t_1) \in R^n$, $\gamma(\cdot) \in L_1([t_1 - h, t_1], R^n)$ entirely determines the essentially constrained trajectory $\psi(t)$ on $[t_0, t_1]$ for known perturbations $v(\cdot) \in L_1([t_0, t_1], R^k)$.

Definition 3.2. The quantity

$$\psi^0_{t_*}(\cdot) = \Big\{\psi(t_*) - A_2'(t_* + h)\psi(t_* + h) + \int_{t_*}^{t_*+h} A_1'(\tau)\psi(\tau)\, d\tau; \tag{3.5}$$

$$\int_{t_*}^{t_*+\theta} A_1'(\tau)\psi(\tau)\, d\tau - A_2'(t_* + \theta)\psi(t_* + \theta), \quad 0 < \theta \leqslant h\Big\}$$

is called the minimal state of (3.3) at the instant of time $t_* \leqslant t_1 - h$. The basic property of (3.5) is that the information $\psi_{t_*}{}^0(\cdot)$, as the prehistory of system (3.3) is minimal (necessary and sufficient) for an unambiguous determination of the trajectory $\psi(t)$ on $(-\infty, t)$ for known perturbations $v(t)$, $t < t_*$, when the condition that the matrices $A(t), A_1(t), A_2(t)$ are supplemented on $(-\infty, t_0)$ while preserving their properties. The following statement is more accurate.

Statement 3.1. Let the matrices $A(t)$, $A_1(t)$, $A_2(t)$ be supplemented on $(-\infty, t_0)$ in such a way that condition d) holds in every finite interval. Then for any $t_* \leqslant t_1 - h$ the realization of the properties $\psi_{t_*}{}^0(\cdot) = 0$ along any arbitrary trajectory of system (3.3) when $v(t) \equiv 0$, $t < t_*$ is equivalent to $\psi(t) \equiv 0$ for almost all $t < t_*$.

Proof. When $v(t) \equiv 0$, $t < t_*$, the conjugate system (3.3) admits of the representation

$$\psi(t) = \psi(t_*) - A_2'(t_* + h)\psi(t_* + h) - \int_{t_*}^{t} [A'(\tau)\psi(\tau) + \tag{3.6}$$

$$A_1'(\tau + h)\psi(\tau + h)]\, d\tau + A_2'(t + h)\psi(t + h), \quad t < t_*$$

By replacing variables we convert (3.6) to the form of the Volterra inhomogeneous equation

$$\psi(t) = -\int_{t_*}^{t} A'(\tau)\psi(\tau)\, d\tau + q(t) \tag{3.7}$$

$$q(t) \overset{\text{def}}{=} \psi(t_*) - A_2'(t_* + h)\psi(t_* + h) + \int_{t_*}^{t_*+h} A_1'(\tau)\psi(\tau)\, d\tau + A_2'(t + h)\psi(t + h) - \int_{t_*}^{t+h} A_1'(\tau)\psi(\tau)\, d\tau, \quad t < t_*$$

It follows from (3.7) that $q(t) \equiv 0$ when $\psi_{t_*}{}^\circ(\cdot) = 0$, and $\psi(t) \equiv 0$ successively in the intervals $[t_* - h, t_*), [t_* - 2h, t_* - h)$ etc. Conversely, if $\psi(t) \equiv 0, t < t_*$, then from (3.7) when $t \in [t_* - 2h, t_* - h)$ it follows that the first component in (3.5) vanishes, and when $t \in [t_* - h, t_*)$ the second component also vanishes for almost all $0 < \theta \leqslant h$.

If the matrix $A_2(t)$ is absolutely continuous in $[t_0, t_1]$, the integral system (3.3) reduces to the difference-differential form

$$\begin{aligned} &\psi^\cdot(t) = -A'(t)\psi(t) - A_1'(t+h)\psi(t+h) + d/dt\,[A_2'(t+\\ &\quad h)\psi(t+h)] + C'(t)v(t), \quad t_0 \leqslant t \leqslant t_1 - h\\ &\psi^\cdot(t) = -A'(t)\psi(t) - \gamma(t) + C'(t)v(t), \quad t_1 - h \leqslant t \leqslant t_1\\ &\psi(t_1) = \psi_1 \in R^n \end{aligned} \tag{3.8}$$

with the conditions of jumps (2.6).

It can be shown that the minimum state (3.5) of system (3.8), (2.6) may be represented in the form

$$\Big\{\psi(t_* - 0);\ A_2'(\tau_{t_*})[\psi(\tau_{t_*} - 0) - \psi(\tau_{t_*} + 0)];\ A_1'(t_* + \theta)\psi(t_* + \theta) - \frac{d}{dt}[A_2'(t_* + \theta)\psi(t_* + \theta)],\ 0 < \theta \leqslant h\Big\}$$

where τ_t is the point of discontinuity of the form $t_1 - ih, i = 1, 2, \ldots$, closest on the right to t.

Let $\Lambda \subset R^n \times L_1([t_1 - h, t_1], R^n)$ be the Banach space of the initial state $\psi_{t_1}(\cdot) = \{\psi(t_1); \gamma(t), t_1 - h \leqslant t \leqslant t_1\}$ of system (3.3) and let $Q \subset L_1([t_0, t_1], R^k)$ be the perturbation space $v(t)$, $t_0 \leqslant t \leqslant t_1$.

Definition 3.3. System (3.3), (3.4) is called the ideally Λ-observable at the instant of time t_0 with perturbations from space Q, if it follows from the condition $z(t) \equiv 0, t_0 \leqslant t \leqslant t_1$ that $\psi_{t_0}{}^\circ(\cdot) = 0$ for any $\psi_{t_1}(\cdot) \in \Lambda$, $v(\cdot) \in Q$.

Remarks. 3.3. If the definition of system (3.3) is extended to $(-\infty, t_0)$ when $v(t) \equiv 0$, $t < t_0$, then by virtue of statement 3.1 the observability is equivalent to the possibility of reconstructing by the output $z(t), t_0 \leqslant t \leqslant t_1$ the trajectory $\psi(t)$ of system (3.3) for almost all $t < t_0$ for any $\psi_{t_1}(\cdot) \in \Lambda$, $v(\cdot) \in Q$.

3.4. The proposed concept of ideal observability of systems with aftereffect was first introduced in /16/ in the case of a system with lag with respect to the state for a specific form of the space Λ, Q (when there are no perturbations, it reduces to observability on the continuation *). In the case of a set of ordinary differential equations, when $D(t) \equiv 0$ the concept introduced is equivalent to ideal observability in the sense of /17/ (see also /3/ and the bibliography given there). If $\psi(t_1) = 0$, the observability is close in spirit to the concept of conditional ideal observability which was introduced and used to solve game problems of dynamics in /3/.

3.5. If $\psi(t_1) = 0$ and the matrix $A_2(t), t_0 \leqslant t \leqslant t_1$ is continuous, the trajectory $\psi(t)$ of system (3.3) is continuous on $[t_0, t_1]$. If then the matrix $A_2(t)$ is absolutely continuous on $[t_0, t_1]$, the respective observability of system (3.8), (2.6) is equivalent to the observability of system (3.8) without the condition of trajectory jumps (2.6), it is then possible to consider the quantity

$$\{\psi(t_*);\ A_1'(t_* + \theta)\psi(t_* + \theta) - d/dt\,[A_2'(t_* + \theta)\psi(t_* + \theta)],\ 0 < \theta \leqslant h\}$$

as the minimum state of system (3.8). It includes the necessary and sufficient information for determining the single-valued absolutely continuous trajectory of system (3.8) on $(-\infty, t_*)$. Note that similar constructions for systems with aftereffect of the type (1.1) are called in /18/ the informer of the solution at the instant t_*.

4. The principle of duality.

Kalman's principle of duality /19/ (see also /1–3/, 19,20 is well-known in the theory of control and observation of conventional linear systems. It established the dual correspondence between the concepts of controllability of the input and the observability of conjugate dynamic systems.

Various duality relations between the controllability and observability of linear systems with lag were obtained in /13, 14, 16, 18, 21/. Note that in /10, 16, 20/ the duality between the controllability and observability of linear systems is derived directly from the duality of correspondence in the theory of extremal convex problems, i.e. Kalman's duality principle is inserted into the general theory of duality of convex analysis. The problems of control are thus reduced to the respective problems of minimizing a functional of the norm type on trajectories of a linear control system with free right-hand end. Below. duality relations are established in this way between the concepts of controllability and observability of systems with aftereffect considered here.

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* Mordukhovich, B.SH. and Sasonkin, A.M., Duality in problems of the control of systems of neutral type with application to controllability and observability. II. Minsk, Deposited at VINITI No. 5267-80, 1980.

Henceforth we will assume that $P = L_p([t_0, t_1], R^k)$, $1 \leqslant p < \infty$, and the norm $\|\cdot\|_\Omega$ in the space of pairs $\{x_1;\ \beta(t),\ t_1 - h \leqslant t \leqslant t_1\}$ is defined by one of the following formulas:

$$1)\ \|(x_1, \beta(\cdot))\|_\Omega = \|x_1\|_{R^u} = |x_1|$$

$$2)\ \|(x_1, \beta(\cdot))\|_\Omega = \|\beta(\cdot)\|_{L_r};\ L_r = L_r([t_1 - h, t_1], R^n),\quad 1 \leqslant r < \infty$$

$$3)\ \|(x_1, \beta(\cdot))\|_\Omega = \|(x_1, \beta(\cdot))\|_{R^u \times L_r} = (|x_1|^r + \|\beta(\cdot)\|_{L_r}{}^r)^{1/r},\quad 1 \leqslant r < \infty$$

Then $P^* = L_q([t_0, t_1], R^k)$, $1/q + 1/p = 1$, and the conjugate space Ω^* has the form

$$1)\ \Omega^* = \{(\psi_1, \gamma(\cdot)) : \psi_1 \in R^n;\ \gamma(t) \equiv 0,\ t_1 - h \leqslant t \leqslant t_1\}$$

$$2)\ \Omega^* = \{(\psi_1, \gamma(\cdot)) : \psi_1 = 0;\ \gamma(\cdot) \in L_s([t_1 - h, t_1], R^n),\quad 1/s + 1/r = 1\}$$

$$3)\ \Omega^* = \{(\psi_1;\ \gamma(\cdot)) : \psi_1 \in R^n;\ \gamma(\cdot) \in L_s([t_1 - h, t_1], R^n),\quad 1/s + 1/r = 1\}$$

Note that in case 1) we may assume in the definition of observability that $\gamma(\cdot)$ is a fixed function from $L_1([t_1 - h, t_1], R^n)$, and in case 2 consider that ψ_1 is a fixed vector from R^n (the condition of jumps (2.6) for system (3.8) is immaterial).

Theorem 4.1. When the assumptions made above are satisfied, the following principle of duality holds. For the Ω-approximate null-controllability of system (1.1), (3.1) on the segment $[t_0, t_1]$ with P-approximations of constraints (3.2) it is necessary and sufficient that the conjugate system (3.3), (3.4) where Ω^* is ideally observable at the instant of time t_0 with perturbations from space P^*.

Proof. Consider case 3) (in other cases the proof by the same scheme is similar). Following the procedure in /16, 20/, we shall formulate the problem of minimizing the functional

$$I(x, u) = |x(t_1)| + \int_{t_1-h}^{t_1} |x(t)|^r\,dt + \int_{t_0}^{t_1} |C(t)x(t) + D(t)u(t)|^p\,dt \to \inf \tag{4.1}$$

on admissible processes of system (1.1), (3.1). Obviously the controllability is equivalent to $\inf I(x, u) = 0$ in problem (1.1), (3.1), (4.1) for any functions $\varphi(t)$ absolutely continuous in (3.1). By virtue of the extremal duality relation (2.7) we have

$$\min J(\psi, w) = -\inf I(x, u) = 0,\ \forall\varphi(\cdot) \tag{4.2}$$

where $\min J(\psi, w)$ which is taken in problem (2.1)–(2.4), which is dual to (1.1), (3.1) and (4.1), and which for $p = 1$, $r = 1$, $w = (v, \gamma)$ can be represented in the form of the problem of minimizing the functional

$$\begin{aligned} J(\psi, v, \gamma) = & \int_{t_0}^{t_0+h} \psi'(t)[A_1(t)\varphi(t-h) + A_2(t)\varphi^{\cdot}(t-h)]\,dt + \\ & \varphi'(t_0)[\psi(t_0) - A_2'(t_0 + h)\psi(t_0 + h)] \to \inf \end{aligned} \tag{4.3}$$

on trajectories of system (3.3) with the constraints

$$B'(t)\psi(t) + B_1'(t + h)\psi(t + h) - D'(t)v(t) \equiv 0,\ t_0 \leqslant t \leqslant t_1 \tag{4.4}$$

$$|\psi(t_1)| \leqslant 1;\ |\gamma(t)| \leqslant 1,\ t_1 - h \leqslant t \leqslant t_1;\ |v(t)| \leqslant 1,\ t_0 \leqslant t \leqslant t_1 \tag{4.5}$$

When $p > 1$, $r > 1$, the functional (4.3) is supplemented by the term

$$\frac{p}{q}\int_{t_0}^{t_1} |v(t)|^q\,dt + \frac{r}{s}\int_{t_1-h}^{t_1} |\gamma(t)|^s\,dt \left(\frac{1}{p} + \frac{1}{q} = 1,\ \frac{1}{r} + \frac{1}{s} = 1\right)$$

and the respective constraints in (4.5) are replaced by

$$v(\cdot) \in L_q([t_0, t_1], R^k),\ \gamma(\cdot) \in L_s([t_1 - h, t_1], R^n) \tag{4.6}$$

which does not alter the essence of subsequent reasoning. Transforming the functional (4.3), taking into account the absolute continuity $\psi(t)$, $t_0 - h \leqslant t \leqslant t_0$, and the formula of integration by parts, we obtain

$$\begin{aligned} J(\psi, v, \gamma) = & \int_{t_0}^{t_0+h} (\varphi^{\cdot}(t-h))'\left[A_2'(t)\psi(t) - \int_{t_0}^{t} A_1'(\tau)\psi(\tau)\,d\tau\right]dt + \\ & \varphi'(t_0)\left[\psi(t_0) - A_2'(t_0 + h)\psi(t_0 + h) + \int_{t_0}^{t_0+h} A_1'(t)\psi(t)\,dt\right] \end{aligned}$$

Hence by virtue of (4.2) we obtain the relations

$$A_2'(t)\psi(t) - \int_{t_0}^{t} A_1'(\tau)\psi(\tau)\,d\tau \equiv 0, \quad t_0 \leqslant t \leqslant t_0 + h$$

$$\psi(t_0) - A_2'(t_0 + h)\psi(t_0 + h) + \int_{t_0}^{t_0+h} A_1'(t)\psi(t)\,dt = 0$$

for any processes $\{\psi(\cdot), v(\cdot), \gamma(\cdot)\}$ admissible in (3.3), (4.4), and (4.6). The latter also means the observability of system (3.3), (3.4) corresponding to case 3). The theorem is proved.

Remarks 4.1. Within the scope of the approach considered here to problems of controllability and observability we can investigate the case when the initial function $\varphi(t)$ in (3.1) is discontinuous when $t = t_0$. Then in Theorem 4.1 the minimal state (3.5) must be replaced in the definition of observability by the quantity

$$\Big\{\psi(t_*) - A_2'(t_* + h)\psi(t_* + h);\ \int_{t_*}^{t_*+h} A_1'(\tau)\psi(\tau)\,d\tau;$$
$$\int_{t_*}^{t_*+\theta} A_1'(\tau)\psi(\tau)\,d\tau - A_2'(t_* + \theta)\psi(t_* + \theta),\ 0 < \theta \leqslant h\Big\}$$

which contains redundant information for calculating the trajectory of system (3.5) in $(-\infty, t_*)$.

4.2. The problem of exact controllability of system (1.1), (3.1) in a given space when the constraints (3.2) are satisfied, reduces in this approach to obtaining the theorems of existence of the optimal controls in the respective optimization problem of the type (1.1), (3.1), (4.1). The class of functions that have a solution is the natural class of (exact) controllability. It also generates linear operations of restitution in the dual problem of observation. The results of /8, 22/ enable general situations to be distinguished, where such problems are solved by passing to generalized pulsed effects /1, 3, 6/.

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